PAIRED ALTERNATIVE ROUTE RELAY-RACES

Eugene V. Larkin[1], *Aleksandr N. Privalov*[2]

[1] Tula State University, Tula
[2] Tula State Pedagogical University named after L.N. Tolstoy, Tula
E-mails: elarkin@mail.ru, privalov.61@mail.ru
Russian Federation

Abstract: It is shown that problem of competition, in which flow of events emerge in physical time, is quite actual. The case, when two partners are performed by teams, every distance is divided onto stages, every stage have alternative routes, and every participant of the team may choice the route to run his stage, is considered. It is shown, that natural approach to modeling of paired alternative route relay-race is two-parallel semi-Markov process with K-arity tree structure. Every brunch of a tree performs an alternative route to overcome the distance. Formulae for calculation of probabilities and time densities, which permit to estimate parameters of running their distance by teams, are obtained. Two arbitrary routes are selected from the structures of two-parallel semi-Markov process and model of the relay-race along selected routes is worked out. The recursive procedure of relay-race evolution analysis with stochastic and time characteristics estimation is worked out. Conception of distributed forfeit, which depends on stages difference of participants, compete in pairs, is proposed. Dependence for evaluation of total forfeit of every participant is obtained. It is shown, that sum of forfeit may be used as optimization criterion in the game strategy optimization task.

Key words: relay-race, concurrent game, two-parallel semi-Markov process, distance, stage, route, evolution, distributed forfeit, recurrent procedure.

1. INTRODUCTION

Relay-races in physical time objectively exist in manifold fields of human activity, such as sport, economics, politics, defense etc. [1, 2, 3, 4, 20]. In classic relay-races a “distance”, that should be overcome by “team” of “participants”, is divided onto “stages”. The “distance running” efficiency depends not only on overall “winning” the whole “distance”, but also on “winning” the “stages” [5] (below terms “distance”, “stage”, “team”, “participant”, “winning”, “forfeit” etc. will be used without inverted commas). In this case both a distance route and stages, in which distance is divided, are determined uniquely, and there is no any alternative to pre-determined route. There is the only parameter, namely the time distribution law, which participant can vary and control. In such a way, there is the trivial solution of winning problem - to run distance stages with the greatest possible speed [6, 7]. This solution is acceptable for the team of participants, who has necessary resources to maintain selected high speed, but if it has not, team tries to find asymmetric response, which leads to the final point of the distance, but along the other route, with alternative division of the route onto stages. For an external observer selection of routes by participants of the team is made randomly, so model of running the distance on alternative routes should be the stochastic one [8, 9]. The availability of alternative generates premises for the emergence of game situations, in which one cane to manage not only by the stage passing time, but also by probabilities (for gamer adversary) or by selection periodicity (for gamer itself) of routs. Those team which can evaluate the benefits and losses from choice of this or that distance route, may construct the optimal strategy of relay-races games for winning the competition as a whole.

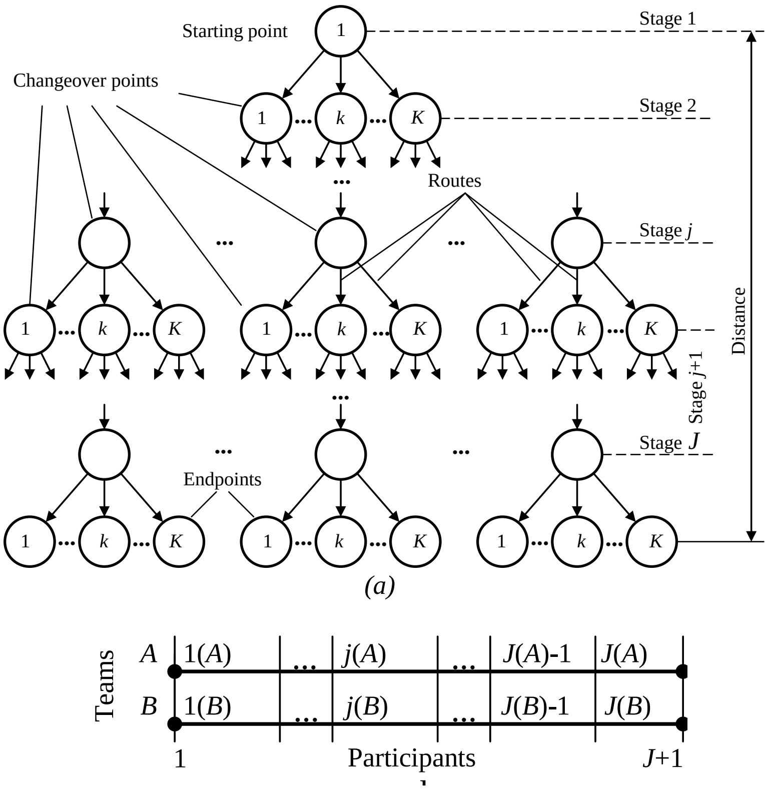
Approaches for forecasting of benefits and losses of relay-races games are currently known insufficiently, that explains necessity and relevance of the investigations in this domain.

2. THE STRUCTURE AND THE MODEL OF PAIRED ALTERNATIVE ROUTE RELAY-RACE

The following assumptions are made below [5]:

- ✓ relay race includes two teams, *A* and *B*, of participants (paired race);
- ✓ two teams act independently of each other;
- ✓ teams *A* and *B* should to overcome the distance in real physical time;
- ✓ the distance is divided into stages, and every stage is overcame by one participant of the team;
- ✓ quantity of stages is equal to quantity of participant in the team;
- ✓ number of the stage is equal the number of participant, who runs the stage;
- ✓ every stage includes *K* routs;
- ✓ after finishing j_A-th stage by j_A-th participant of team *A* (j_A+1)-th participant may choose one of *M* possible routes of the (j_A+1)-th stage;
- ✓ after finishing j_B-th stage by j_B-th participant of team *B* (j_B+1)-th participant may choose one of *K* possible routes of the (j_B+1)-th stage;
- ✓ participants with number 1 of both teams, *A* and *B*, start one of the first stage possible routes at once;
- ✓ time of passing of every stage route by the participant is a random one and is defined individually for him with accuracy to density;
- ✓ after completion of current stage by previous participant next participant starts next stage without a lag;
- ✓ winning or losing of a stage competition is understood as completion the stage the first or not the first;
- ✓ winner's forfeit is distributed in time and depends on difference of stages and routes, which pass winner and loser.

Distance, stages and routes are shown on fig. 1 a, where next indexes are used: $1 \le j \le J$ - numbers of stage, which coincide with number of participant of proper team; $1 \le k \le K$ - numbers of stages routs after the changeover points. As it is shown on the fig 1 a, the structure of distance is represented with *K*-arity tree of *J*+1 hierarchical levels. On the first hierarchical level there is starting point of competition. On the (*J*+1)-th hierarchical level there are endpoints of competition. Competition as such is shown on the fig. 1 b. Point 1 performs the starting one and point *J*+1 performs the endpoint of competition. Indexes of participants are performed as functions. Bracketed part of index points to the team.



(b)

Fig. 1. Distances, stages (a) and participants (b) of relay-race

Model of paired alternative route relay-race may be performed as two-parallel semi-Markov process

$$\boldsymbol{h}(t)=\begin{bmatrix}\boldsymbol{h}_A(t) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{h}_B(t)\end{bmatrix}, \qquad (1)$$

where t - is the time; $\boldsymbol{h}_A(t)$, $\boldsymbol{h}_B(t)$ - are semi-Markov matrices [8. 10. 11] of size $\frac{K^{J+1}-1}{K-1}\times\frac{K^{J+1}-1}{K-1}$, which describe passing the routes of distance by teams A and B correspondingly; $\boldsymbol{0}$ - is the zero matrix of size $\frac{K^{J+1}-1}{K-1}\times\frac{K^{J+1}-1}{K-1}$;

$$\boldsymbol{h}_{A,B}(t)=\left[h_{m(A,B),n(A,B)}(t)\right]; \qquad (2)$$

$h_{m(A,B),n(A,B)}(t)=f_{m(A,B),n(A,B)}(t)p_{m(A,B),n(A,B)}$ - is the weighted time density of residence semi-Markov process in the state $m(A, B)$ with further switching to the state $n(A, B)$; $f_{m(A,B),n(A,B)}(t)$ - is the pure time density of residence the semi-Markov process in the state $m(A,B)$ with further switching to the state $n(A, B)$; $p_{m(A,B),n(A,B)}$ - is the probability of switching from the state $m(A,B)$ to the state $n(A, B)$;

$$\int_0^\infty h_{m(A,B),n(A,B)}(t)dt = p_{m(A,B),n(A,B)}; \qquad (3)$$

$$f_{m(A,B),n(A,B)}(t) = \frac{h_{m(A,B),n(A,B)}(t)}{p_{m(A,B),n(A,B)}}. \qquad (4)$$

States $1(A)$, $1(B)$ are the starting ones. States

$$\frac{K^{J(A,B)}-1}{K-1} < n(A,B) \le \frac{K^{J(A,B)+1}-1}{K-1} \qquad (5)$$

are the absorbing ones. When semi-Markov process switches into absorbing state, it stops. Structure, shown on the fig. 1 a, permits to describe semi-Markov matrices $\boldsymbol{h}_A(t)$ and $\boldsymbol{h}_B(t)$. If one numerates states of semi-Markov process top-down, and left-to-right, then in rows with indices $1 \le m(A,B) \le \frac{K^{J(A,B)}-1}{K-1}$ elements $2 + K[m(A,B)-1] \le n(A,B) \le 1 + Km(A,B)$ are non-zero ones. All other element of semi-Markov matrices are zeros.

For non-zero rows:

$$\sum_{n(A,B)=1}^{M(A,B)} p_{m(A,B),n(A,B)} = 1, \qquad (6)$$

where $M(A,B) = \frac{K^{J(A,B)+1}-1}{K-1}$.

The tree structure permits to evaluate time and probabilities of absorbing states achievement from the starting state in $J(A,B)$ switches [14]:

$$h_{n(A,B)}(t) = L^{-1}\left[{}^r\boldsymbol{I}_1 \cdot \{L[\boldsymbol{h}_{A,B}(t)]\}^{J(A,B)c} \cdot \boldsymbol{I}_{n(A,B)}\right]; \qquad (7)$$

$$p_{n(A,B)} = \int_0^\infty h_{n(A,B)}(t)dt; \qquad (8)$$

$$f_{n(A,B)}(t) = \frac{h_{n(A,B)}(t)}{p_{n(A,B)}}, \qquad (9)$$

where $n(A,B)$ satisfies the inequality (5); ${}^r\boldsymbol{I}_1$ - is the row vector of size $\frac{K^{J(A,B)+1}-1}{K-1}$, first element of which is equal to one, and other elements are equal to zeros; $\boldsymbol{I}_{n(A,B)}$ - is the column vector of size $\frac{K^{J(A,B)+1}-1}{K-1}$, $[n(A,B)]$-th element of which is equal to one, and other elements are equal to zeros; $L[...]$ and $L^{-1}[...]$ - are direct and inverse Laplace transform correspondingly.

To simplify indexing one would change index $m(A,B)$ or $n(A,B)$ to the index $i(A,B)$ as follows:

$$i(A,B) = m(A,B) - \frac{K^{J(A,B)} - 1}{K-1}, \qquad (10)$$

and numerate with indices $i(A,B)$ route realizations of wondering through the trees of Markov processes $\boldsymbol{h}_A(t)$ and $\boldsymbol{h}_B(t)$. Emergence of all possible routes constitutes full group of incompatible events, so

$$\sum_{i(A,B)=1}^{K^{J(A,B)}} p_{i(A,B)} = 1. \qquad (11)$$

Full time of wandering through the semi-Markov processes as a whole is as follows:

$$f_{A,B}(t) = \sum_{i(A,B)}^{K^{J(A,B)}} h_{i(A,B)}(t) \qquad (12)$$

3. COMPETITION IN PAIRED ALTERNATIVE ROUTE RELAY-RACES

Let us extract from tree *A* and tree *B* route realizations *i*(*A*) and *i*(*B*) (in common, *i*(*A*) ≠ *i*(*B*)) and create sets of states, included to routs, correspondingly

$$S_A = \{1[i[A]], \ldots, j[i[A]], \ldots, J[i[A]]\};$$
$$S_B = \{1[i[B]], \ldots, j[i[B]], \ldots, J[i[B]]\}. \qquad (13)$$

Cartesian product of sets gives space *S* [5], in which one coordinate is the number of element in the set S_A, and other coordinate is the number of element in the set S_B:

$$S = S_A \times S_B = \{(1[i(A)], 1[i(B)]), \ldots, (j[i(A)], j[i(B)]), \ldots, (J[i(A)], J[i(B)])\}. \qquad (14)$$

Space is shown on the fig. 2 a.

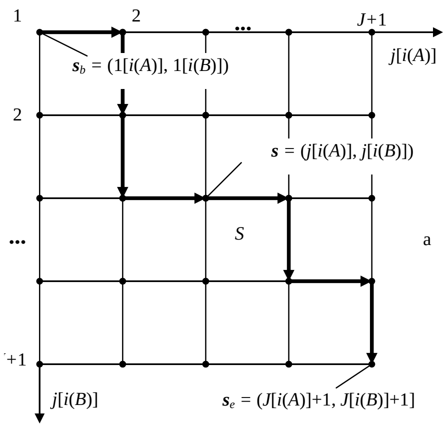
Common number of states in the space *S* is equal to $(J+1)^2$. Initial meaning of vector is $\boldsymbol{s}_b = (1, 1)$. Wandering through the space has the character of evolution, in which after every switch one of the vector ***s*** elements increase by unit. At any switches incremental element is the only one. Switches last till vector ***s*** will reach the state $\boldsymbol{s}_e = (J, J)$. Common number of switches during evolution is as 2*J*.

Evolution has the tree structure, example of which for the case *J* = 3 is shown on the fig. 2 b. Nodes of graph are marked with two digits. First digit means number of the stage, which run participant from the team *A*, second digit means number of the stage, which run participant from the team *B*. When there is the possibility for increment both stage numbers, there are two arcs, which issue from the node. When one of the team finishes the last stage of its distance, only one arc issues from the node. Dash line on the fig. 2 b divides the evolution tree onto two symmetrical parts.

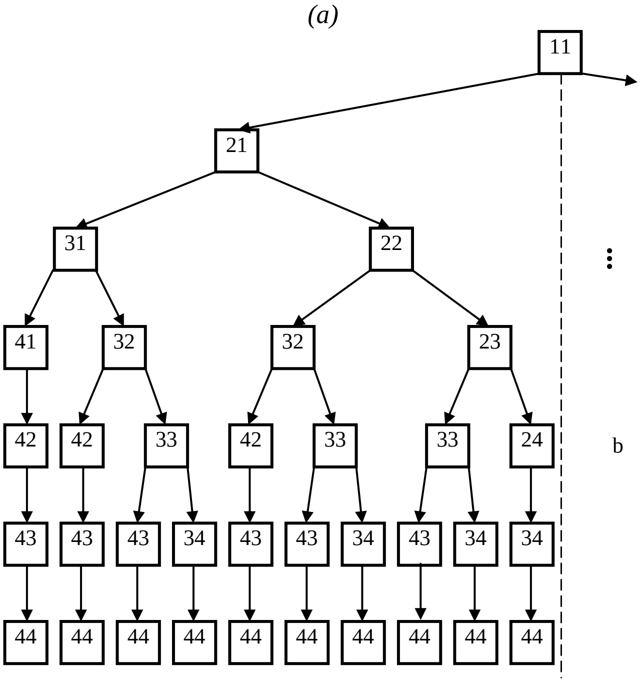
During evolution route realizations *i*(*A*) and *i*(*B*) compete between them [5, 9]. In common case, when compete two participants, who starts simultaneously and run their stages during time, definable with densities $\theta_A(t)$ and $\theta_B(t)$, then time of completion the stage the first is as follows

$$\theta_w(t) = \theta_A(t)[1 - \Theta_B(t)] + \theta_B(t)[1 - \Theta_A(t)] = \theta_{wA}(t) + \theta_{wB}(t), \qquad (15)$$

where $\theta_{wA}(t)$ - is the weighted density of time of winning the stage by participant *A*; $\theta_{wB}(t)$ - is the weighted density of time of winning the stage by participant A; $\Theta_{...}(t)=\int_{0}^{t}\theta_{...}(\tau)d\tau$ - distribution function.



(a)



(b)

Fig. 2. Space S of states (a) and tree of switches (b)

Probability of winning participants *A* and *B*, and pure time densities of such events are as follows:

$$\pi_{wA} = \int_0^\infty \theta_{wA}(t)dt; \quad \pi_{wB} = \int_0^\infty \theta_{wB}(t)dt;. \quad (16)$$

$$\varphi_{wA} = \frac{\theta_{wA}(t)}{\pi_{wA}}; \quad \varphi_{wB} = \frac{\theta_{wB}(t)}{\pi_{wB}}. \quad (17)$$

Second formula, necessary for relay-races simulation, is the dependence for waiting time density. If from competing participants $\theta_A(t)$ and $\theta_B(t)$ wins the participant $\theta_B(t)$, he waits until $\theta_A(t)$ complete his stage. Formula for density of waiting time is as follows [3, 5]

$$\theta_{B\to A}(t) = \frac{\eta(t)\int_0^\infty \theta_B(\xi)\theta_A(t+\xi)d\xi}{\int_0^\infty \Theta_B(t)d\Theta_A(t)}, \quad (18)$$

where $\eta(t)$ - is the Heaviside function; ξ - is additional argument having the dimension of time; $\Theta_{A,B}(t) = \int_0^t \theta_{A,B}(\xi)d\xi$.

With use of dependences (16), (17), (18) may be formed recursive procedure of relay-race evolution analysis. Evolution of relay-race develops as competition in *S* space. Let us introduce the concept of switching trajectory in the space *S*, shown on the fig 2 a with bold arrows. This trajectory shows the realization of stochastic sequence switches of vector $\boldsymbol{s}$ from state $\boldsymbol{s}_b$, till state $\boldsymbol{s}_e$. A trajectory as a whole is an occasional one, but the realization of a trajectory is quite deterministic for every concrete case of evolution [15, 16].

Let us consider some (f.e. *l*-th) case of evolution, which starts from the state $\boldsymbol{s}_b = (1[i(A)], 1[i(A)])$. When there are no switches $r = 0$, participants pass the first stages of their distances and in competition participate initial densities from first rows of matrices $\boldsymbol{h}_A(t)$, $\boldsymbol{h}_B(t)$.

Let us in *l*-th realization under consideration participant from the team *B* wins, and vector $\boldsymbol{s}_b$ is transformed to $\boldsymbol{s} = (1[i(A)], 2[i(A)])$. Probability of winning by the participant from the team *B* may be obtained from (16) [13], as follows:

$$\pi_{i(A),i(B),l,1} = \int_0^\infty {}^0g_{1[i(B)]}(t)\left[1 - {}^0G_{1i(A)}(t)\right]dt, \quad (19)$$

where ${}^{\cdots}G_{\ldots}(t) - \int_0^t {}^{\cdots}g_{\ldots}(\tau)d\tau$ - is the distribution function; ${}^0g_{1[i(A)]}(t) = f_{1[i(A)]}(t)$; ${}^0g_{1[i(B)]}(t) = f_{1[i(B)]}(t)$; $f_{1[i(A)]}(t)$, $f_{1[i(B)]}(t)$ - are time densities of first stage running of the distance routes *i*(*A*) and *i*(*B*), correspondingly; upper left index points to number of previous switches; lower right index points to number of stage, in which participant stays at the current moment of time.

Density of time, winner spent for running the first stage, is equal to [5]

$$\varphi_{i(A),i(B),l,1}(t) = \frac{{}^0g_{1[i(B)]}(t)\left[1 - {}^0G_{1[i(A)]}(t)\right]}{\pi_{i(A),i(B),l,1}}. \quad (20)$$

After winning by participant from the team *B* the first stage, second participant from the team *B* begins to run the second stage, although first participant from the team *A* continues running his first stage. This is why after first switch

$$^{1}g_{2[i(A)]}(t):={}^{0}g_{1[i(B)]\to 1[i(A)]}(t);\ {}^{1}g_{2[i(B)]}(t):=f_{2[i(B)]}(t), \qquad (21)$$

where ":=" - is the operation of substitution; $^{0}g_{1(B)\to 1(A)}(t)$ is determined as (18)

$$^{1}g_{1[i(B)]\to 1[i(A)]}(t)=\frac{\eta(t)\int_0^\infty {}^{1}g_{1[i(B)]}(\xi)\cdot{}^{1}g_{1[i(A)]}(t+\xi)d\xi}{\int_0^\infty {}^{1}G_{1[i(B)]}(t)d\,{}^{1}G_{1[i(A)]}(t)}. \qquad (22)$$

After the first switch densities $^{1}g_{2[i(A)]}(t)$ and $^{1}g_{2[i(B)]}(t)$ begin to compete. For the concrete *l*-th case of evolution, they are evaluated as (21).

Let after *r* switches in the *l*-th case of evolution vector ***s*** = (*j*[*i*(*A*)], *j*[*i*(*B*)]), densities $^{r}g_{j[i(A)]}(t)$ and $^{r}g_{j[i(B)]}(t)$ compete between them, in competition wins the team *A* participant and vector ***s*** transforms to ***s*** = (*j*[*i*(*A*)] +1, *j*[*i*(*B*)]).

Probability of his winning may be obtained from (16) [13], as follows:

$$\pi_{i(A),i(B),l,r}=\int_0^\infty {}^{r}g_{j[i(A)]}(t)\left[1-{}^{r}G_{j[i(B)]}(t)\right]dt\,. \qquad (23)$$

where $^{r}g_{j[i(A)]}(t)$ and $^{r}g_{j[i(B)]}(t)$ are time densities, obtained as a result of the *l*-th case of evolution.

Density of time, winner spent for running the current stage, is equal to [5]

$$\varphi_{i(A),i(B),l,r}(t)=\frac{{}^{r}g_{j[i(A)]}(t)\left[1-{}^{r}G_{j[i(B)]}(t)\right]}{\pi_{i(A),i(B),l,r}}. \qquad (24)$$

After winning by *j*[*i*(*A*)]-th participant from the team *A* the *j*(*A*)-th current stage, (*j*[*i*(*A*)]+1)-th participant from the same team begins to run the next stage, although *j*[*i*(*B*)]-th participant from the team *B* continues his current stage. This is why after *r* + 1 switches

$$^{r+1}g_{j[i(B)]}(t):={}^{r}g_{j[i(A)]\to j[i(B)]}(t);\ {}^{r+1}g_{j[i(A)]+1}(t):=f_{j[i(A)]+1}(t), \qquad (25)$$

where ":=" - is the operation of substitution;

$$^{r}g_{j[i(A)]\to j[i(B)]}(t)=\frac{\eta(t)\int_0^\infty {}^{r}g_{j[i(A)]}(\xi)\cdot{}^{r}g_{j[i(B)]}(t+\xi)d\xi}{\int_0^\infty {}^{r}G_{j[i(A)]}(t)d\,{}^{r}G_{j[i(B)]}(t)}. \qquad (26)$$

After the *r*+1 switches densities $^{r+1}g_{j[i(B)]}(t)$ and $^{r+1}g_{j[i(A)]+1}(t)$ begin to compete. For the concrete *l*-th case of evolution, they are evaluated as (25).

Let after *r* = 2*J* - 1 switches in the *l*-th case of evolution vector ***s*** = (*J*[*i*(*A*)], *J*[*i*(*B*)]+1). There is no competition at all on this stage, participant from the team *A* finish his last stage alone in time, which is defined as ${}^{2J-1}g_{J[i(A)]}(t)$, and vector $\boldsymbol{s}$ is switched to $\boldsymbol{s}_e = (J[i(A)]+1, J[i(B)]+1)$.

Let us note, that during evolution may emerge the situation, in which $\boldsymbol{s} = (J[i(A)]+1, j[i(B)])$, or $\boldsymbol{s} = (j[i(A)], J[i(B)]+1)$ In this case there is no competition of teams, and all next substitutions are as follows:

$$ {}^{J[i(A)]+j[i(B)]+1}g_{j[i(B)]+1}(t) = \varphi_{i(A),i(B),l,r}(t) = f_{j[i(B)]+1}(t),\ r = J[i(A)] + j[i(B)] + 1; $$

$$ {}^{J[i(B)]+j[i(A)]+1}g_{j[i(A)]+1}(t) = \varphi_{i(A),i(B),l,r} = f_{j[i(A)]+1}(t);\ r = J[i(B)] + j[i(A)] + 1. \qquad (27) $$

Common quantity of realizations $\widetilde{L}$, of evolution trajectories with Indices $1 \le l \le \widetilde{L}$ grows fast in dependence of number of stages $J(A, B)$ (R. Bellman's "curse of dimensionality" [17]). Restricts the growth of realizations quantity those fact that during switches some participants get $[J+1]$-th state and further evolution on this branch is impossible (fig. 2 b).

Probability of l-th realization of evolution and density of time of finishing of l-th realization of relay-race are as follows:

$$ \pi_{i(A),i(B),l} = \prod_{r=0}^{2J+1} \pi_{i(A),i(B),l,r} \qquad (28) $$

$$ \varphi_{i(A),i(B),l}(t) = L^{-1} \prod_{r=0}^{2J+1} L\left[\varphi_{i(A),i(B),l,r}(t)\right], \qquad (29) $$

where $L[\ldots]$ и $L^{-1}[\ldots]$ - are direct and inverse Laplace transforms correspondingly; $\pi_{l,r} \in \{\pi_{l,r,A}, \pi_{l,r,B}\}$; $\varphi_{l,r}(t) \in \{\varphi_{l,r,A}(t), \varphi_{l,r,B}(t)\}$..

All possible issues of stage competitions constitutes full group of incompatible events, so

$$ \sum_l \pi_{i(A),i(B),l} = 1; $$

$$ \sum_l \pi_{i(A),i(B),l} \cdot \varphi_{i(A),i(B),l}(t) = f_{i(A),i(B)}(t), \qquad (30) $$

where $f_{i(A),i(B)}(t)$ - is density of mean time of completion of competition on the routes $i(A), i(B)$.

4. EVALUATION OF EFFECTIVENESS OF ALTERNATIVE RELAY-RACE STRATEGY

Quite natural for evaluation of effectiveness is the model, in which:

all possible pairs of routes from the starting points of semi-Markov processes $\boldsymbol{h}_A(t)$, $\boldsymbol{h}_B(t)$ till the absorbing states which numbers defined with inequality (5), are under consideration;

into pair one route of team A, and another route of team B are included;

forfeit amount is defined as distributed payment $c_{m(A),n(B)}(t)$ in $\left[\frac{\text{doll}}{\text{time}}\right]$, value of which depends on time (f.e. on installed principle "the longer, the more expensive", "the longer, the more cheaper", or in simplest case- time independent forfeit);

if $c_{m(A),n(B)}(t)$>0, then the team *A* acquires from the team *B*, a forfeit;

if $c_{m(A),n(B)}(t)$<0, then the team *B* acquires from the team *A*, a forfeit;

if $c_{m(A),n(B)}(t)$=0, then no one pays anyone.

Elements $c_{m(A),n(B)}(t)$ are stacked into global forfeit matrix of size $\frac{K^{J+1}-1}{K-1} \times \frac{K^{J+1}-1}{K-1}$:

$$\boldsymbol{c}(t) = \left[c_{m(A),n(B)}(t)\right],\ 1 \le m(A),\ n(B) \le \frac{K^{J+1}-1}{K-1}. \qquad (31)$$

Due to features the process under consideration matrix (31) is quite asymmetrical. From the global matrix (31), with taken into account that competition evolution develops on the routes *i*(*A*) and *i*(*B*), may be formed matrix ${}^{i(A),i(B)}\boldsymbol{c}(t)$ by means of selections of proper elements from the global matrix $\boldsymbol{c}(t)$, and recalculation of indices in accordance with (10):

$$\boldsymbol{c}(t) = {}^{i(A),i(B)}\boldsymbol{c}(t) = \left[c_{j[i(A)],\, j[i(B)]}\right]. \qquad (32)$$

Analysis of evolution shows that situation, which change conditions of forfeit payments emerges from moment of winning the previous stage by participant of one of team and lasts till the moment of next outcome of current competition. So, if compete routes *i*(*A*) and *i*(*B*) and evolution develops on the *l*-th scenario, considered above, then after *r* switches common sum of forfeit on the current stage is as follows [3]:

$${}^{l,r}C_{j[i(A)],\, j[i(B)]} = \int_0^\infty c_{j[i(A)],\, j[i(B)]}(t) \cdot \varphi_{i(A),i(B),l,r}(t)dt, \qquad (33)$$

where $\varphi_{i(A),i(B),l,r}(t)$ is defined as (24).

Sums of forfeit of the sequential stages are summed, so

$${}^{l}C_{i(A),i(B)} = \sum_{r=1}^{2J+1} {}^{l,r}C_{j[i(A)],\, j[i(B)]}, \qquad (34)$$

where part of indices, namely *j*[...], is omitted, due index *l* contain it.

This forfeit is paid with probability, $\pi_{i(A),i(B),l}$ which is defined with (28).

In common, forfeit sum of relay-race along the selected pair of routes *i*(*A*), *i*(*B*) is as follows:

$$C_{ji(A),i(B)} = \sum_l \pi_{i(A),i(B),l} \cdot {}^{l}C_{i(A),i(B)}. \qquad (35)$$

Common cost of the game is defined by means of weighed summing of costs $C_{ji(A),i(B)}$ (35) as follows:

$$C_\Sigma = \sum_{i(A)=1}^{K^{J(A)}} \sum_{i(B)}^{K^{J(B)}} p_{j(A)} \cdot p_{j(B)} \cdot C_{j(A),\, j(B)}, \qquad (36)$$

where $p_{j(A)}$, $p_{j(B)}$ probabilities of choice of routes, which are defined as (8) with indices recalculated as (10).

Formula (36) may be used as a criterion of optimization of strategy of alternative route relay-race games. As optimization variables in this case may be used parameters of initial semi-Markov process (1), such as probabilities $p_{m(A,B),n(A,B)}$ (3) and parameters of time densities $f_{m(A,B),n(A,B)}(t)$.

5. CONCLUSION

Working out of model of alternative route relay-race and usage the formula for estimation cost of the strategy opens new page in the game theory because competition develops in real physical time, and on the every stage of game there is alternative for choice of route for continuation the competition. Moreover, on every stage, as in real life, there is race between participants of different team, in which any participant can win or lose by time, that amplifies a competitive moment of game under consideration. Usage the conception of two-parallel semi-Markov process permits to work out the simple approach to modeling alternative route relay-race with simply and naturally estimated parameters of teams, participated the game.

Further investigation in this area should be directed to finding more tight links with classical game theory and use typical optimal strategies in dynamic competition condition [18, 19]. Also it is possible of working out radically new strategies, oriented only on the use with the model of paired alternative route relay-races.

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Information about the authors:

Larkin Eugene, Doctor of Technical Sciences, full professor, Tula State University, Head. Russian Federation. Areas of scientific research – data processing, semi-Markov process, computer modeling, Petri-Markov networks.

Privalov Aleksandr, Doctor of Science (Tech.), full professor, Tula State Pedagogical University “L.N. Tolstoy”, Russian Federation. Areas of scientific research - data processing, semi-Markov process, computer modeling, Petri-Markov networks

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