MATHEMATICAL MODEL WRITTEN BY ORDINARY DIFFERENTIAL EQUATIONS FOR HYSTERESIS GAMES

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Abstract: In game theory, the hysteresis effect manifests itself in the fact that small differences in one or more parameters lead two systems to opposite stable equilibrium. The mathematical model (called smooth model) of hysteresis game writing by ordinary differential equations with the great parameter K is studied. Estimations of closeness of output functions for smooth and classical models through the continuity module of continuous input function are received. This error can be controlled by increasing the parameter K to a sufficiently large value. The result was conducted by studying of mathematical model and using programs Mathematica, Matlab or Maple. Experiments were carried out, and the obtained results were summarized and presented through evaluations and graphical interpretation.

Key words: game theory, hysteresis, mathematical model, smooth model, continuity module.

1. INTRODUCTION

In game theory, the hysteresis effect manifests itself in the fact that small differences in one or more parameters bring two systems into an opposite stable equilibrium, for example, the "good" ones — trust, honesty and a high level of well—being; and the "bad" ones - theft, distrust, corruption and poverty. Despite the small initial differences, systems require huge efforts to move from one equilibrium state to another. Hysteresis games are mathematically modeled using a nonlinear differential equation. Its solution is more complicated than the solution of an ordinary differential equation, which is called a smooth model. Therefore, the study of a smooth model is a significant novelty. The authors' contribution consists in proving the use of a smooth model to obtain an approximate solution to the classical model. Further, it is shown that the smooth model offers a good tool for numerical analysis using information technology applications, such as Mathematica, Matlab, Maple, etc. The simulation results show that the smooth model proposed by us can replace the classical model.

2. MATHEMATICAL MODEL OF HYSTERESIS GAMES

As we know, study of mathematical models is the tool to understand characteristics of interesting problems [1–6]. One of the problems is to research by differential equations a mathematical model of hysteresis games [7, 8]. Hysteresis games are operators, inputs of which are continuous functions. At firsts, we give a basic description of games [7]. A game on an interval $[0,1]$ is an operator, the output function of which is $g(s)=\max\{g_0,x(s)\}$ and $g(s)=\min\{g_0,x(s)\}$ to a monotonically increasing and decreasing, respectively, continuous input function $x(s)$ and an initial condition $g_0\in[x(s_0),x(s_0)+1]$.

Let us recall the classic model for hysteresis games [8], an output function $y(s)$ of which is defined by

$$\dot{y}(s)=\begin{cases}\max\{\dot{\sigma}(s),0\} & \text{if } y(s)=\sigma(s),\\ 0 & \text{if } \sigma(s)<y(s)<\sigma(s)+1,\\ \min\{\dot{\sigma}(s),0\} & \text{if } y(s)=\sigma(s)+1.\end{cases} \tag{1}$$

In [8] it is studied that, the differential equation (1) has an uniquely solution $y(s)$ under some initial conditions $y_0\in[\sigma(s_0),\sigma(s_0)+1]$. This function is called a output. Here the output function $y(s)$ is a locally absolutely continuous and the Lipschitz constant of this output $y(s)$ is 1.

In game theory, the hysteresis effect manifests itself in the fact that small differences in one or more parameters lead two systems into opposite stable equilibrium, for example, "good" — trust, honesty and high welfare; and "bad" — theft, distrust, corruption and poverty. Despite the small initial differences, systems require huge efforts to move from one equilibrium to another. Interesting applications are given in [9, 10, 11].

The aim of this paper is to show, that we research a new model in the form of an ordinary differential equation involving a great control parameter M. We prove, that we can be replace classic model (1) to new model successfully. The main contribution of this study is to estimate the error between solution of classic model (1) and solution of the new model. We check that an arbitrary small error can be obtained by increasing the great control parameter M.

2. THE SMOOTH MODEL OF GAMES

Let us $x(s)$ is a continuous function on $[s_0,s_0+S]$, the piecewise smooth function $\sigma(s)$ is defined as follows:

$$\begin{cases}\dot{\sigma}=M\left[x(s)-x\left(s-\dfrac{1}{M}\right)\right]\\ \sigma(s_0)=x(s_0)=x_0.\end{cases} \tag{2}$$

Now, we study another model of games which we call *smooth model.* Let $x(s)$ is a continuous function, we define the corresponding output $\eta = \eta(s)$ which is defined by

$$\dot{\eta}(s) = \frac{\max\{\sigma(s) - \eta(s), 0\} - \max\{\eta(s) - 1 - \sigma(s), 0\}}{M}, \tag{3}$$

here M is a great parameter.

In order to measure the small error between the output functions of the classic model (2) and the smooth model (3), we consider a *smooth* output $\overline{\eta}(s)$ of the game for a smooth input a smooth input $\overline{x}(s)$ on $[s_0, s_0 + S]$ as follows

$$\dot{\overline{\eta}}(s) = \frac{\max\{\overline{x}(s) - \overline{\eta}(s), 0\} - \max\{\overline{\eta}(s) - 1 - \overline{x}(s), 0\}}{M} \tag{4}$$

Suppose $\overline{y}(s)$ is an output of the game satisfying (1) which corresponds to smooth input $\overline{x}(s)$ and we denote $C := \sup\{|\dot{\overline{x}}(s)| : s_0 \le s \le s_0 + S\}$.

Theorem 1.

Suppose the output function $\overline{y}(s)$ *of classic model* (1) *and the output function* $\overline{\eta}(s)$ *of smooth model* (4) *satisfy some initial condition*

$$\overline{y}(s_0) = \overline{\eta}(s_0) = g_0 \in [x(s_0), x(s_0) + 1]$$

Then, we can measure the error between these output functions as follows

$$|\overline{y}(s) - \overline{\eta}(s)| \le \frac{C}{M} \text{ for } s \in [s_0, s_0 + S]. \tag{5}$$

Proof. This is easily see that the estimate (5) is true for $C = 0$. So suppose that $C > 0$ and the estimate (5) is not true. Then, there is a moment $\tau_0 \in [s_0, s_0 + S]$ and for some $\delta > 0$ such that

$$|\overline{y}(\tau_0) - \overline{\eta}(\tau_0)| = \frac{C}{M} \tag{6}$$

$$|\overline{y}(s) - \overline{\eta}(s)| > \frac{C}{M}. \tag{7}$$

At first, if $|\overline{y}(\tau_0) - \overline{\eta}(\tau_0)| = \overline{y}(\tau_0) - \overline{\eta}(\tau_0) = \frac{C}{M}$, then

$$\overline{y}(s) - \overline{\eta}(s) > \frac{C}{M} \text{ for } s \in (\tau_0, \tau_0 + \delta), \tag{8}$$

since outputs $\overline{y}(s)$ and $\overline{\eta}(s)$ are continuous functions. We prove that in every point $s \in (\tau_0, \tau_0 + \delta)$, we have

$$(\overline{y}(s) - \overline{\eta}(s))' \le 0. \tag{9}$$

By virtue of (8) we get $\overline{\eta}(s) < \overline{x}(s) + 1$ for $s \in (\tau_0, \tau_0 + \delta)$. If $\overline{\eta}(s) \ge \overline{x}(s)$, then it follows $\overline{x}(s) < \overline{y}(s) \le \overline{x}(s) + 1$. By virtue of (1) and (4) we obtain

$$\left(\overline{y}(s)-\overline{\eta}(s)\right)' = \begin{cases} 0 & \text{if } \overline{y}(s)\in\left(\overline{x}(s),\overline{x}(s)+1\right), \\ \min\{0,\dot{\overline{x}}\} & \text{if } \overline{y}(s)=\overline{x}(s)+1. \end{cases} \tag{10}$$

This easily follows (9).

Next, if $\overline{\eta}(s)<\overline{x}(s)$ for some s, then (1) and (4) follow that

$$\left(\overline{y}(s)-\overline{\eta}(s)\right)' = \begin{cases} -M\left(\overline{x}(s)-\overline{\eta}(s)\right) & \text{if } \overline{y}(s)\in\left(\overline{x}(s),\overline{x}(s)+1\right) \\ \max\{0,\dot{\overline{x}}\}-M\left(\overline{x}(s)-\overline{\eta}(s)\right) & \text{if } \overline{y}(s)=\overline{x}(s) \\ \min\{0,\dot{\overline{x}}\}-M\left(\overline{x}(s)-\overline{\eta}(s)\right) & \text{if } \overline{y}(s)=\overline{x}(s)+1. \end{cases} \tag{11}$$

It easily see that, the estimate (9) is true in the case $\overline{x}(s)<\overline{y}(s)\le\overline{x}(s)+1$. In case $\overline{y}(s)=\overline{x}(s)$, we check that

$$\max\{0,\dot{\overline{x}}\}-M\left(\overline{x}(s)-\overline{\eta}(s)\right)<\max\{0,\dot{\overline{x}}\}-C\le 0.$$

because $\overline{x}(s)-\overline{\eta}(s)>\frac{C}{M}$ (see (8)).

Thus, the estimate (9) is true. Using (6) and (9) we get

$$\overline{y}(s)-\overline{\eta}(s)\le\overline{y}(\tau_0)-\overline{\eta}(\tau_0)=\frac{C}{M} \text{ for } s\in(\tau_0,\tau_0+\delta)$$

which contradicts (7).

Similarly, we can prove the true of (9) in case, when $\overline{y}(\tau_0)-\overline{\eta}(\tau_0)=-\frac{C}{M}$. The proof of the Theorem 1 is completed.

Now we estimate the error between solutions of the classic model (1) and the smooth model (3) whose input is continuous function.

Theorem 2. *Suppose that* $x(s)$ *is a continuous function on* $[s_0,s_0+S]$; $y(s)$ *and* $\eta(s)$ *are solutions of* (1) *and* (3) *which are satisfied to the initial conditions*

$$y(s_0)=\eta(s_0)=g_0\in\left[x(s_0),x(s_0)+1\right].$$

Then, we have for every $M>0$ *the estimate*

$$\left|\eta(s)-y(s)\right|\le 2\varepsilon\left(x,\frac{1}{M}\right) \text{ for } s\in[s_0,s_0+S], \tag{12}$$

where

$$\varepsilon(x,\delta):=\sup\left\{\left|x(s_1)-x(s_2)\right|:\left|s_1-s_2\right|\le\delta \text{ and } s_1,s_2\in[s_0,s_0+S]\right\}. \tag{13}$$

Proof. Let M_1, M_2 are great values. We denote by $\sigma_1(s)$, $\sigma_2(s)$ piecewise smooth functions are defined by (2), in which the great parameter M is M_1 and M_2, respectively. Let $y_1(s),y_2(s)$ are solutions of (1) with x replaced by $\sigma_1(s),\sigma_2(s)$, respectively. Moreover, denote by $\eta_1(s),\eta_2(s)$ solutions of (3) with inputs $\sigma_1(s),\sigma_2(s)$ and parameters $M=M_1,M=M_2$, respectively. Suppose that output functions $y_j(s)$ and $\eta_j(s)$ satisfy the initial condition

$$y_j(s_0) = \eta_j(s_0) = g_0 \in \left[x(s_0), x(s_0)+1\right] \ (j=1,2).$$

Using the result of the Theorem 1 we have

$$\left|y_j(s) - \eta_j(s)\right| \le \frac{1}{M_j} \sup\left\{\left|\dot{\sigma}_j(s)\right| : \mathrm{s}_0 \le s \le s_0 + S\right\}. \qquad (14)$$

Consequently,

$$\left|y_j(s) - \eta_j(s)\right| \le \varepsilon\left(x, \frac{1}{M_j}\right) \quad \text{for } s \in \left[s_0, s_0 + S\right] \qquad (15)$$

In [8] it is show that the Lipschitz constant of the output game of classic model (1) is 1, that is

$$\left|y_1(s) - y_2(s)\right| \le \left|\sigma_1(s) - \sigma_2(s)\right| \ \text{ for } s \in \left[s_0, s_0 + S\right] \qquad (16)$$

By virtue of (2) we get

$$\sigma_j(\mathrm{s}) = x(\mathrm{s}_0) + M_j \int_{s_0}^{s} x(\tau)d\tau - M_j \int_{s_0 - 1/\mathrm{M}_j}^{s - 1/\mathrm{M}_j} x(\tau)d\tau = M_i \int_{s - 1/\mathrm{M}_j}^{s} x(\tau)d\tau$$

It follows that

$$\sigma_i(s) - x(s) = M_j \int_{s - 1/\mathrm{M}_j}^{s} [x(\tau) - x(s)]d\tau \ \text{ for } s \in \left[s_0, s_0 + S\right].$$

This implies that

$$\left|\sigma_j(s) - x(s)\right| \le \varepsilon\left(x, \frac{1}{M_j}\right) \ \text{for } s \in \left[s_0, s_0 + S\right]. \qquad (17)$$

So from (16) and (17) we follow that

$$\left|y_1(s) - y_2(s)\right| \le \varepsilon\left(x, \frac{1}{M_1}\right) + \varepsilon\left(x, \frac{1}{M_2}\right). \qquad (18)$$

Next, we have

$$\left|\eta_1(s) - y_2(s)\right| \le \left|\eta_1(s) - y_1(s)\right| + \left|y_1(s) - y_2(s)\right|. \qquad (19)$$

Now we choose $M = M_1$ and $M_2 \to +\infty$; then by virtue of (17) we get $\sigma_2(s) \underset{M_2 \to +\infty}{\rightrightarrows} x(s)$ on $\left[s_0, s_0 + S\right]$, while $y_2(s) \underset{M_2 \to +\infty}{\to} y(s)$. Finally, from (15), (18) and (19) this implies that the estimate (12) is true, and the proof of the Theorem 2 is completed.

3. NUMERICAL ANALYSIS AND NEARNESS ESTIMATES

To analysis results of the paper we consider some numerical examples. Using the Mathematica program [12] we show trajectories of the classic model (1) by sketching graphical solution of the smooth model (3). Let we take

$$x(s) := -\frac{\left|s \sin s - 1\right|}{2} + 2; \quad g_0 := 0{,}8; \quad M = M_1 := 10^6$$

Then, the solution of the smooth model (3) is shown in Figure 1.

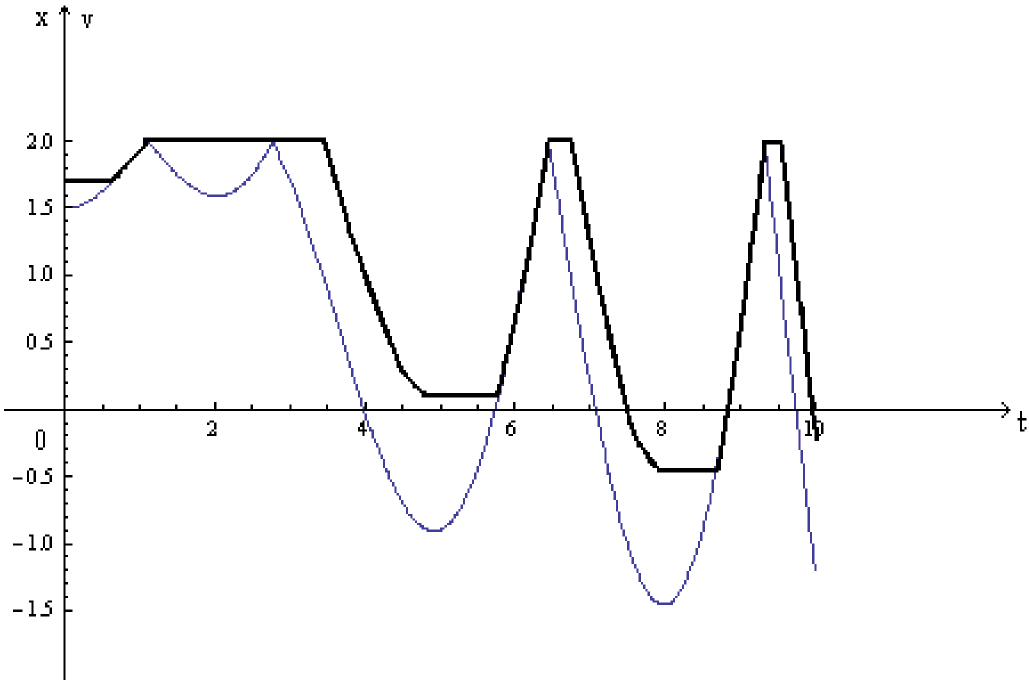

Figure 1. Trajectory of the game for sufficiently great parameter M

Using the Mathematica program we get the following estimate

$$\varepsilon\left(x, \frac{1}{M}\right) \le 10^{-5}$$

on $[s_0, s_0 + S] = [0,10]$.

Using results of the Theorem 2 we have

$$|\eta(s) - y(s)| \le \frac{2}{10^6} \text{ on } [0,10].$$

If we take another large parameter $M = \frac{M_1}{2}$, then the solution of (3) is the same. This means that the new model (3) of the game is stable and does not depend on parameter M if M is sufficiently great value.

4. CONCLUSION

In game theory, the hysteresis effect manifests itself in the fact that small differences in one or more parameters lead two systems into opposite stable equilibrium, for example, "good" — trust, honesty and high welfare; and "bad" — theft, distrust, corruption and poverty. Despite the small initial differences, systems require huge efforts to move from one equilibrium to another. Hysteresis games have been mathematically modelled by the nonlinear differential equation (1). Obviously, solving the equation (1) is more difficult than solving the ordinary differential equation (3) which is called the smooth model. Therefore, studying the smooth model (3) has high scientific interest. Firstly, we can use the smooth model (3) to obtain the approximated solution of the classic model (1). Secondly, the smooth model offers a good tool for numerical analysis using applied programs of information technology, example Mathematica, Matlab, Maple, etc. Simulation results demonstrate that our proposed smooth model can replace the classical model. Based on this paper, we hope to take more good impacts in further studies.

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